A PSEUDODIFFERENTIAL HÖRMANDER'S INEQUALITY

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Abstract. The classical Hörmander's inequality for linear partial differential operators with constant coefficients is extended to pseudodifferential operators.

1. Introduction

The classical Hörmander's inequality proved in [3] for linear partial differential operators with constant complex coefficients $P(D)$ and a bounded domain Ω of $\mathbb{R}^n$ claims that

$$\left\|\left(\partial^{\alpha} P\right)(D)\, u\right\|_0 \leq c \left\|P\left(D\right) u\right\|_0 \ , \forall u \in C_0^{\infty}\left(\Omega\right)$$

where $\alpha \in \mathbb{N}^n$ and $c > 0$ is independent of u.

In the proof, we obviously see that in fact the following result hold

$$\forall \delta > 0, \exists \rho > 0, \forall u \in C_0^{\infty}\left(\Omega\right), diam(\Omega) < \rho,$$

$$\left\|\left(\partial^{\alpha} P\right)(D)\, u\right\|_0 \leq \delta \left\|P\left(D\right) u\right\|_0$$

The aim of this paper is to give an extension of this Hörmander's inequality to classical pseudodifferential operators with constant coefficients. The proposed pseudodifferential Hörmander's inequality includes the given cases in [5] and [1]. The formulation of our result is the following theorem.

Theorem 1. *Let $P\left(D\right)$ be a pseudodifferential operator of the class S^m, and let $s \in \mathbb{R}, \theta \geq 1$ and $\alpha \in \mathbb{N}^n$, then*

$$\forall \delta > 0, \exists \rho > 0, \exists c > 0, \forall u \in C_0^{\infty}\left(\Omega\right), diam(\Omega) < \rho,$$

$$\left\|\left(\partial^{\alpha} P\right)(D)\, u\right\|_s \leq \delta \left\|P\left(D\right) u\right\|_s + c \left\|u\right\|_{s+m-\theta}$$

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2. Preliminaries

The set Ω denotes an open domain of $\mathbb{R}^n$ and $\mathbb{N}$ the set of natural numbers $\{1,2,...\}$. The notations and classical definitions from the theory of distributions and pseudodifferential operators are given in [4] and [7]; in particular, $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions with compact support and $H^s, s\in\mathbb{R}$, is the Sobolev space on $\mathbb{R}^n$ with scalar product and norm denoted, respectively, $(\cdot,\cdot)_s$ and $\|\cdot\|_s$.

We will use the following classical inequalities, $s,t\in\mathbb{R}$,

$$|(u,v)_s|\leq \|u\|_{s-t}\|v\|_{s+t} \ , \forall u\in H^{s-t}, \forall v\in H^{s+t} \ ,$$

and

Lemma 1. *Let $\varphi\in C_0^\infty(\Omega)$ and $s\in\mathbb{R}$, then there exists a constant $c>0$ such that*

$$\|\varphi u\|_s\leq \max_x|\varphi(x)|\,\|u\|_s+c\,\|u\|_\sigma \ , \forall u\in H^s \ ,$$

where $\sigma<s-1$.

Proof. See lemma 2 of [6] □

The class of symbols of pseudodifferential operators with constant coefficients is defined as follows.

Definition 1. *The class S^m is the space of infinitely differentiable functions $P(\xi)$ defined on $\mathbb{R}^n$ and satisfying $\forall\alpha\in\mathbb{Z}_+^n$, there exists $c>0$ such that*

$$|(\partial^\alpha P)(\xi)|\leq c(1+|\xi|)^{m-|\alpha|} \ , \forall\xi\in\mathbb{R}^n$$

A pseudodifferential operator $P(D)$ of order m with constant coefficients is an operator acting on functions $u\in H^s$ by the formula

$$P(D)u=\int_{\mathbb{R}^n}e^{ix\cdot\xi}P(\xi)\,\widehat{u}(\xi)d\xi$$

where $P(\xi)\in S^m$ is called the symbol of $P(D)$, and $\widehat{u}$ denotes the Fourier transform of u .

The pseudodifferential operators $(\partial^\alpha P)(D)$, $\alpha\in\mathbb{N}^n$, and $\overline{P}(D)$ are the pseudodifferential operators with respective symbols $(\partial^\alpha P)(\xi)$ and $\overline{P(\xi)}$. It is easy to see that the H^s-adjoint operator of $P(D)$ is the operator $\overline{P}(D)$, and we have

$$\left\|\overline{P}u\right\|_s=\|Pu\|_s \ , \forall u\in H^s$$

Remark 1. *Our a priori estimates are local and in view of a classical result of the theory of pseudodifferential operators, all the operators considered in this work are properly supported.*

By Ω_ε we denote the open ball of center the origin and radius $\varepsilon > 0$. Let $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 1$ for $|x| \leq 1, 0 \leq \varphi(x) \leq 1$ and $\varphi(x) = 0$ for $|x| > 2$. Define $\varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon})$ and let $P(D)$ be a pseudodifferential operator of the class S^m . The operator $[P(D), \varphi_\varepsilon]$ denotes the commutator of the pseudodifferential operator $P(D)$ and the operator of multiplication by the function $\varphi_\varepsilon(x)$.

Lemma 2. *If $0 < \rho < \varepsilon$, then the operator $[P(D), \varphi_\varepsilon]$ is of infinite order in $C_0^\infty(\Omega_\rho)$, i.e. for every reals s and s', there exists $c > 0$ such that*

$$\|[P(D), \varphi_\varepsilon]u\|_s \leq c\|u\|_{s'} \quad , \forall u \in C_0^\infty(\Omega_\rho)$$

Proof. It is deduced from the fact that the symbol of the pseudodifferential operator $[P(D), \varphi_\varepsilon]$ is identically equals zero on a neighbourhood of the open set Ω_ρ □

Remark 2. *We will apply the following algebraic inequality,*

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2 \ , \forall \varepsilon > 0, \forall a \geq 0, \forall b \geq 0$$

3. The inequality

The principal result of the paper is the following theorem which is an extension of Hörmander's inequality.

Theorem 2. *Let $P(D)$ be a pseudodifferential operator of the class S^m, and let $s \in \mathbb{R}, \theta \geq 1$ and $\alpha \in \mathbb{N}^n$, then*

$$\forall \delta > 0, \exists \rho > 0, \exists c > 0, \forall u \in C_0^\infty(\Omega), diam(\Omega) < \rho,$$

$$\|(\partial^\alpha P)(D)u\|_s \leq \delta\|P(D)u\|_s + c\|u\|_{s+m-\theta} \tag{3.1}$$

Proof. Without lost of generality let Ω_ε be the open ball of center the origin and radius $\varepsilon > 0$. Let $\varphi(x) \in C_0^\infty(\mathbb{R}^n), \varphi(x) = 1$ for $|x| \leq 1, 0 \leq \varphi(x) \leq 1$ and $\varphi(x) = 0$ for $|x| > 2$. Define $\varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon})$, then $\varphi(x) = 1$ for $|x| \leq \varepsilon$ and $\varphi_\varepsilon(x) = 0$ for $|x| > 2\varepsilon$, so if $0 < \rho < \varepsilon$, we will have

$$u = \varphi_\varepsilon u \quad , \forall u \in C_0^\infty(\Omega_\rho)$$

Let ∂_j^k denotes the derivation of order k with respect to the variable ξ_j. It is well-known from the theory of pseudodifferential operators that

$$P(ix_j u) = ix_j Pu + (\partial_j P)u \ , \tag{3.2}$$

so, $\forall u \in C_0^\infty(\Omega_\rho)$, with $0 < \rho < \varepsilon$, we have

$$P(ix_j u) = ix_j \varphi_\varepsilon(x) Pu + (\partial_j P)u + T_1 u \ , \tag{3.3}$$

where

$$T_1 = ix_j[P, \varphi_\varepsilon]$$

Then

$$\|(\partial_j P)\, u\|_s^2 = (P\,(ix_j u)\,, (\partial_j P)\, u)_s - (ix_j \varphi_\varepsilon\,(x)\, Pu, (\partial_j P)\, u)_s - (T_1 u, (\partial_j P)\, u)_s$$

It is easy to see that

$$(P\,(ix_j u)\,, (\partial_j P)\, u)_s = \left(\overline{(\partial_j P)}\,(ix_j u)\,, \overline{P} u\right)_s \ ,$$

and consequently, we obtain

$$\|(\partial_j P)\, u\|_s^2 = \left(\overline{(\partial_j P)}\,(ix_j u)\,, \overline{P} u\right)_s - (ix_j \varphi_\varepsilon\,(x)\, Pu, (\partial_j P)\, u)_s - (T_1 u, (\partial_j P)\, u)_s$$

From (3.3), we have

$$\overline{(\partial_j P)}\,(ix_j u) = ix_j \varphi_\varepsilon\,(x)\, \overline{(\partial_j P)} u + \overline{\left(\partial_j^2 P\right)} u + T_2 u \ , \tag{3.4}$$

where

$$T_2 = ix_j[\overline{(\partial_j P)}, \varphi_\varepsilon]$$

Consequently, we have the following inequality

$$\begin{aligned} \|(\partial_j P)\, u\|_s^2 \ \leq\ & \left\| ix_j \varphi_\varepsilon\,(x)\, \overline{(\partial_j P)} u \right\|_s \|Pu\|_s + \left\| \left(\partial_j^2 P\right) u \right\|_s \|Pu\|_s \\ & + \|ix_j \varphi_\varepsilon\,(x)\, Pu\|_s \,\|(\partial_j P)\, u\|_s + \left| \left(T_2 u, \overline{P} u\right)_s \right| + \left| (T_1 u, (\partial_j P)\, u)_s \right| \end{aligned} \tag{3.5}$$

The lemma 2 gives

$$\begin{aligned} \left\| ix_j \varphi_\varepsilon\,(x)\, \overline{(\partial_j P)} u \right\|_s \ &\leq\ \max_x |ix_j \varphi_\varepsilon\,(x)| \left\| \overline{(\partial_j P)} u \right\|_s + c_{s,\sigma}\,(\varepsilon) \left\| \overline{(\partial_j P)} u \right\|_\sigma \\ &\leq\ 2\varepsilon \,\|(\partial_j P)\, u\|_s + c_{s,\sigma}\,(\varepsilon)\, \|u\|_{\sigma+m-1} \ , \ \sigma < s-1 \ , \end{aligned}$$

and

$$\begin{aligned} \|ix_j \varphi_\varepsilon\,(x)\, Pu\|_s \ &\leq\ \max_x |ix_j \varphi_\varepsilon\,(x)|\, \|Pu\|_s + c'_{s,\sigma}\,(\varepsilon)\, \|Pu\|_\sigma \\ &\leq\ 2\varepsilon \,\|Pu\|_s + c'_{s,\sigma}\,(\varepsilon)\, \|u\|_{\sigma+m} \ , \ \sigma < s-1 \end{aligned}$$

For every real t, we have

$$\left| \left(T_2 u, \overline{P} u\right)_s \right| = |(PT_2 u, u)_s| \leq \|T_2 u\|_{s-t+m} \,\|u\|_{s+t} \ ,$$

and

$$\left| (T_1 u, (\partial_j P)\, u)_s \right| = \left| \left(\left(\overline{\partial_j P}\right) T_1 u, u \right)_s \right| \leq \|T_1 u\|_{s-t+m-1} \,\|u\|_{s+t}$$

The above inequalities are resumed in the following one

$$\begin{aligned}
\left\|(\partial_j P)u\right\|_s^2 &\le 4\varepsilon \left\|Pu\right\|_s \left\|(\partial_j P)u\right\|_s + \left\|\left(\partial_j^2 P\right)u\right\|_s \left\|Pu\right\|_s + c'_{s,\sigma}(\varepsilon)\left\|(\partial_j P)u\right\|_s \left\|u\right\|_{\sigma+m} + \\
&\quad + c_{s,\sigma}(\varepsilon)\left\|Pu\right\|_s \left\|u\right\|_{\sigma+m-1} + \left\|T_1 u\right\|_{s-\tau+m-1}\left\|u\right\|_{s+\tau} + \left\|T_2 u\right\|_{s-t+m}\left\|u\right\|_{s+t} \\
&\le 6\varepsilon \left\|Pu\right\|_s^2 + 4\varepsilon \left\|(\partial_j P)u\right\|_s^2 + \frac{1}{8\varepsilon}\left\|\left(\partial_j^2 P\right)u\right\|_s^2 + \frac{\left[c_{s,\sigma}(\varepsilon)\right]^2}{8\varepsilon}\left\|u\right\|_{\sigma+m-1}^2 + \\
&\quad + \frac{\left[c'_{s,\sigma}(\varepsilon)\right]^2}{8\varepsilon}\left\|u\right\|_{\sigma+m}^2 + \left\|u\right\|_{s+t}^2 + \frac{1}{2}\left\|T_1 u\right\|_{s-t+m-1}^2 + \frac{1}{2}\left\|T_2 u\right\|_{s-t+m}^2
\end{aligned} \tag{3.6}$$

Due to the lemma 5, the operators T_1 et T_2 are of infinite orders. Let $\varepsilon > 0$ with $1 - 4\varepsilon > 0$ and let $\sigma = s - \theta,\ \theta > 1$, then there exists a constant $c_{s,\theta}(\varepsilon) > 0$ such that

$$\left\|(\partial_j P)u\right\|_s^2 \le \frac{6\varepsilon}{1-4\varepsilon}\left\|Pu\right\|_s^2 + \frac{1}{(1-4\varepsilon)8\varepsilon}\left\|\left(\partial_j^2 P\right)u\right\|_s^2 + c_{s,\theta}(\varepsilon)\left\|u\right\|_{s+m-\theta}^2$$

Let $\delta > 0$ and take $\varepsilon = \dfrac{\delta}{2(2\delta+3)}$, then $\forall s \in \mathbb{R}, \forall \theta > 1, \forall \delta > 0$, there exist $c_1(\delta) > 0$ and $c_{1,s,\theta}(\delta) > 0$,

$$\left\|(\partial_j P)u\right\|_s^2 \le \delta \left\|Pu\right\|_s^2 + c_1(\delta)\left\|\left(\partial_j^2 P\right)u\right\|_s^2 + c_{1,s,\theta}(\delta)\left\|u\right\|_{s+m-\theta}^2 \tag{3.7}$$

$\forall u \in C_0^\infty(\Omega_\rho), \rho < \varepsilon \le \varepsilon_1(\delta) = \dfrac{\delta}{2(2\delta+3)}$.

Let us show by induction that $\forall k \ge 1, \forall s \in \mathbb{R}, \forall \theta > 1, \forall \delta > 0$, there exist $c_k(\delta) > 0, c_{k,s,\theta}(\delta) > 0$ and $\varepsilon_k(\delta) > 0, \forall u \in C_0^\infty(\Omega_\rho), \rho < \varepsilon \le \varepsilon_k(\delta)$, we have

$$\left\|\left(\partial_j^k P\right)u\right\|_s^2 \le \delta \left\|Pu\right\|_s^2 + c_k(\delta)\left\|\left(\partial_j^{k+1} P\right)u\right\|_s^2 + c_{k,s,\theta}(\delta)\left\|u\right\|_{s+m-\theta}^2 \tag{3.8}$$

The case $k = 1$ is true by (3.7). Assume $\forall l \le k-1, \forall s \in \mathbb{R}, \forall \theta > 1, \forall \delta_l > 0$, there exist $c_l(\delta_l) > 0, c_{l,s,\theta}(\delta_l) > 0, \forall u \in C_0^\infty(\Omega_\rho), \rho < \varepsilon \le \varepsilon_l(\delta_l)$, we have

$$\left\|\left(\partial_j^l P\right)u\right\|_s^2 \le \delta_l \left\|Pu\right\|_s^2 + c_l(\delta_l)\left\|\left(\partial_j^{l+1} P\right)u\right\|_s^2 + c_{l,s,\theta}(\delta_l)\left\|u\right\|_{s+m-\theta}^2 \tag{3.9}$$

Apply the inequality (3.7) to the operator $\left(\partial_j^{k-1} P\right)$, i.e. $\forall\ \delta' > 0$, there exist positive constants $c_1(\delta'), c_{1,s,\theta}(\delta')$ and $\varepsilon_1(\delta'), \forall u \in C_0^\infty(\Omega_\rho), \rho < \varepsilon \le \varepsilon_1(\delta')$, we have

$$\left\|\left(\partial_j^k P\right)u\right\|_s^2 \le \delta' \left\|\left(\partial_j^{k-1} P\right)u\right\|_s^2 + c_1(\delta')\left\|\left(\partial_j^{k+1} P\right)u\right\|_s^2 + c_{1,s,\theta}(\delta')\left\|u\right\|_{s+m-\theta}^2 \tag{3.10}$$

In (3.10), we estimate $\left\|\left(\partial_j^{k-1}P\right)u\right\|_s^2$ by the inequality (3.9) with $l = k-1$, then $\forall u \in C_0^\infty(\Omega_\rho), \rho < \varepsilon \le \min\{\varepsilon_1(\delta'), \varepsilon_{k-1}(\delta_{k-1})\}$, we obtain

$$\begin{aligned}\left\|\left(\partial_j^{k}P\right)u\right\|_s^2 &\le \delta'\delta_{k-1}\|Pu\|_s^2 + \delta' c_{k-1}(\delta_{k-1})\left\|\left(\partial_j^{k}P\right)u\right\|_s^2 + \\ &\quad + \delta' c_{k-1,s,\theta}(\delta_{k-1})\|u\|_{s+m-\theta}^2 + c_1(\delta')\left\|\left(\partial_j^{k+1}P\right)u\right\|_s^2 + c_{1,s}(\delta')\|u\|_{s+m-\theta}^2\end{aligned}$$

Choose $\delta' < \dfrac{1}{c_{k-1}(\delta_{k-1})}$, so $\forall u \in C_0^\infty(\Omega_\rho), \rho < \varepsilon \le \min\{\varepsilon_1(\delta'), \varepsilon_{k-1}(\delta_{k-1})\}$, we have

$$\begin{aligned}\left\|\left(\partial_j^{k}P\right)u\right\|_s^2 &\le \frac{\delta'\delta_{k-1}}{1-\delta' c_{k-1}(\delta_{k-1})}\|Pu\|_s^2 + \frac{c_1(\delta')}{1-\delta' c_{k-1}(\delta_{k-1})}\left\|\left(\partial_j^{k+1}P\right)u\right\|_s^2 + \\ &\quad + \frac{c_{1,s}(\delta') + \delta' c_{k-1,s,\theta}(\delta_{k-1})}{1-\delta' c_{k-1}(\delta_{k-1})}\|u\|_{s+m-\theta}^2\end{aligned}$$

Let $\delta > 0$, take $\delta' = \dfrac{\delta}{\delta_{k-1} + \delta c_{k-1}(\delta_{k-1})}$ and

$$c_k(\delta_k) = \frac{c_1(\delta')}{1-\delta' c_{k-1}(\delta_{k-1})},$$

and

$$c_{k,s}(\delta_k) = \frac{c_{1,s}(\delta') + \delta' c_{k-1,s}(\delta_{k-1})}{1-\delta' c_{k-1}(\delta_{k-1})}$$

Then, we obtain

$$\left\|\left(\partial_j^{k}P\right)u\right\|_s^2 \le \delta_k\|Pu\|_s^2 + c_k(\delta_k)\left\|\left(\partial_j^{k+1}P\right)u\right\|_s^2 + c_{k,s,\theta}(\delta_k)\|u\|_{s+m-\theta}^2$$

$\forall u \in C_0^\infty(\Omega_\rho), \rho < \varepsilon \le \varepsilon_k(\delta_k) = \min\left\{\varepsilon_1\left(\dfrac{\delta}{\delta_{k-1} + \delta c_{k-1}(\delta_{k-1})}\right), \varepsilon_{k-1}(\delta_k)\right\}.$

We have proved the inequality (3.9) for $l = k$. So the estimate (3.8) is true.

Let $\delta_k, c_k(\delta_k), c_{k,s,\theta}(\delta_k)$ et $\varepsilon_k(\delta_k), k = 1, 2, ..., l$, be the respective constants of the right member of the estimates (3.8). Iterating these inequalities, then $\forall u \in C_0^\infty(\Omega_\rho), \rho < \varepsilon \le \min\{\varepsilon_1(\delta_1), ..., \varepsilon_l(\delta_l)\}, \forall\, l \ge 1$, we obtain the following one,

$$\begin{aligned}\left\|\left(\partial_j P\right)u\right\|_s^2 &\le (\delta_1 + c_1(\delta_1)\delta_2 + ... + c_1(\delta_1)c_2(\delta_2)...c_{l-1}(\delta_{l-1})\delta_l)\|Pu\|_s^2 + \\ &\quad + c_1(\delta_1)c_2(\delta_2)....c_l(\delta_l)\left\|\left(\partial_j^{l+1}P\right)u\right\|_s^2 + \\ &\quad (c_{1,s,\theta}(\delta_1) + c_{2,s,\theta}(\delta_2)c_1(\delta_1) + c_{l,s,\theta}(\delta_l)c_1(\delta_1)...c_{l-1}(\delta_{l-1}))\|u\|_{s+m-\theta}^2\end{aligned} \tag{3.11}$$

Let $\delta > 0$, choose $\delta_1, ..., \delta_l$ from the following equations

$$\delta_1 = \frac{\delta}{l}, c_1(\delta_1)\delta_2 = \frac{\delta}{l}, ..., c_1(\delta_1)c_2(\delta_2)...c_{l-1}(\delta_{l-1})\delta_l = \frac{\delta}{l},$$

and define the constants $c_l(\delta)$ and $c_{l,s,\theta}(\delta)$ respectively as the coefficients of the terms $\left\|\left(\partial_j^{t+1}P\right)u\right\|_s^2$ and $\|u\|_{s+m-\theta}^2$ in the inequality (3.11), then $\forall l\geq 1, \forall s\in\mathbb{R}, \forall\theta>1, \forall\delta>0$, there exist $c_l(\delta)>0, c_{l,s,\theta}(\delta)>0$ and $\varepsilon_l(\delta)>0$, $\forall u\in C_0^\infty(\Omega_\rho), \rho<\varepsilon\leq\varepsilon_l(\delta)$, we have

$$\|(\partial_jP)u\|_s^2\leq\delta\|Pu\|_s^2+c_l(\delta)\left\|\left(\partial_j^{l+1}P\right)u\right\|_s^2+c_{l,s,\theta}(\delta)\|u\|_{s+m-\theta}^2 \tag{3.12}$$

Choose $l\in\mathbb{N}$ with $l\geq\theta-1>0$, then there is $c_{s,\theta}(\delta)>0, \forall u\in C_0^\infty(\Omega_\rho), \rho<\varepsilon\leq\min\{\varepsilon_1(\delta_1),...,\varepsilon_l(\delta_l)\}$,

$$\|(\partial_jP)u\|_s^2\leq\delta\|Pu\|_s^2+c_{s,\theta}(\delta)\|u\|_{s+m-\theta}^2 \tag{3.13}$$

The inequality (3.13) is true for $\theta=1$, because the operator (∂_jP) is of order $s+m-1$. Finally, we have proved that $\forall s\in\mathbb{R}, \forall\theta\geq1, \forall\delta>0, \exists\rho>0, \exists c>0, \forall u\in C_0^\infty(\Omega), diam(\Omega)<\rho$,

$$\|(\partial_jP)u\|_s^2\leq\delta\|Pu\|_s^2+c\|u\|_{s+m-\theta}^2 \tag{3.14}$$

Let $\alpha=(\alpha_1,...,\alpha_n)$ and $\alpha'=(\alpha_1,...,\alpha_{j-1},\alpha_j+1,\alpha_{j+1}...,\alpha_n)$ be given multi-indices. Assume as an hypothesis of induction : $\forall s\in\mathbb{R}, \forall\theta\geq1, \forall\delta>0, \forall\alpha\in\mathbb{Z}_+^n, \exists\rho>0, \exists c>0, \forall u\in C_0^\infty(\Omega), diam(\Omega)<\rho$,

$$\left\|\left(\partial_j^\alpha P\right)u\right\|_s^2\leq\delta\|Pu\|_s^2+c\|u\|_{s+m-\theta}^2 \quad , \tag{3.15}$$

is true. Apply the inequality (3.14) to the operator $(\partial^\alpha P)$, then we have

$$\left\|\left(\partial_j^{\alpha'}P\right)u\right\|_s^2\leq\delta'\|(\partial^\alpha P)u\|_s^2+c'\|u\|_{s+m-\theta}^2\ ,\ u\in C_0^\infty(\Omega')\quad ,$$

where Ω' depends on δ'. From the hypothesis of induction for the operator$(\partial^\alpha P)$, we obtain that for every $\delta>0$, there is $\rho>0$ such that

$$\left\|\left(\partial_j^{\alpha'}P\right)u\right\|_s^2\leq\delta'\delta\|Pu\|_s^2+\delta'c\|u\|_{s+m-\theta}^2+c'\|u\|_{s+m-\theta}^2\quad ,$$

$u\in C_0^\infty(\Omega\cap\Omega'), diam(\Omega)<\rho$. Let $\gamma>0$, choose $\delta'=\dfrac{\gamma}{\delta}$, we obtain then the inequality (3.15) for α'. This ends the proof of the theorem $\square$

Remark 3. *As the operator $(\partial^\alpha P)$ is of order $m-|\alpha|$, the estimate (3.1) is trivial for $\theta\leq|\alpha|$, the theorem is then restated in the following way :*
Let $P(D)$ be a pseudodifferential operator of the class $S^m, s', s\in\mathbb{R}$, and $\alpha\in\mathbb{N}^n$, $s'<s+m-|\alpha|$, then $\forall\delta>0, \exists\rho>0, \exists c>0, \forall u\in C_0^\infty(\Omega), diam(\Omega)<\rho$,

$$\|(\partial^\alpha P)(D)u\|_s\leq\delta\|P(D)u\|_s+c\|u\|_{s'}\quad ,$$

In this form, the theorem reminds the so called Erhling's inequality.

References

[1] Bouzar C., Kuleshov A. A., Local solvability of pseudodifferential equations of constant strength I. Differential Equations 24, no. 5, 548–553, (1988).

[2] Bouzar C., Local estimates for pseudodifferential operators. Dokl. Nats. Akad. Nauk Belarusi 44, no. 4, 18–20, (2000)

[3] Hörmander L., On the theory of general partial differential operators. Acta Math., 94, 161-248 (1955).

[4] Hörmander L., The analysis of linear partial differential operators I. Springer, Second Edition 1990.

[5] Paneah B., Pseudodifferential operators of constant strength in the main. Math. Sbornik 2:2, 179-198, (1967).

[6] Peetre J., A proof of the hypoellipticity of formally hypoelliptic differential operators. Comm. Pure Appl. Math., 14, 737-744, (1961).

[7] Treves F., Introduction to pseudodifferential and Fourier integral operatoprs, Volume I, Plenum Press, 1982.

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